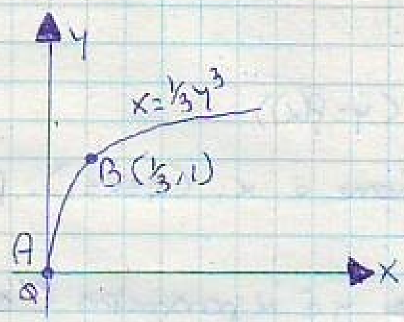
dove Γ è l'arco appartenente alla parabola cubica di equazione $x = \frac{1}{3}y^3$ congiungente i punti $A(0,0)$ e $B(\frac{1}{3},1)$.



In questo tipo di esercizi le equazioni parametriche della linea non vengono date esplicitamente, come per l'esercizio precedente, ma vengono fornite implicitamente dall'equazione della linea Γ:

$$x = \frac{1}{3}y^3.$$

Questa è una funzione del tipo:

$$\boxed{x = f(y)}$$

In questo caso il parametro da prendere in considerazione è y.

$$0 \le y \le 1 \Rightarrow ds = \sqrt{x'^2(y)+1} = \sqrt{y^4+1}\, dy$$

Dunque:

$$I = \int_0^1 y^7 \{1+(3x)^{4/3}\}^2 \sqrt{y^4+1}\, dy = \int_0^1 y^7 \{1+y^4\}^2 \sqrt{y^4+1}\, dy =$$

$$= \int_0^1 y^7 (y^4+1)^{5/2} dy$$

Posto: $\sqrt{1+y^4} = t \Rightarrow t^2 = 1+y^4 \Rightarrow t^2-1 = y^4 \Rightarrow 4y^3 dy = 2t \Rightarrow y^3 dy = \frac{1}{2} t\, dt$

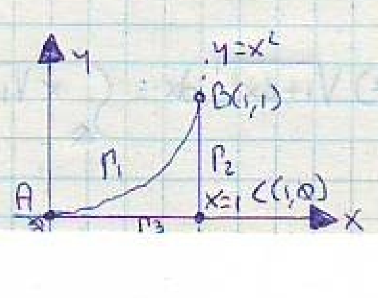
Dunque:

$$I = \int_0^1 y^7 (1+y^4)^{5/2} dy = \frac{1}{2}\int_1^{\sqrt{2}} (t^2-1)\, t^6\, dt = \frac{1}{2}\left[\frac{t^9}{9} - \frac{t^7}{7}\right]_1^{\sqrt{2}} = \frac{1}{2}\left(\frac{2^{9/2}}{9} - \frac{2^{7/2}}{7} + \right.$$

$$\left. - \frac{1}{9} + \frac{1}{7}\right) = \frac{1}{2} 2^{7/2}\left(\frac{2}{9} - \frac{1}{7}\right) + \frac{1}{63} = 4\sqrt{2}\, \frac{5}{63} + \frac{1}{63} = \frac{1}{63}(20\sqrt{2}+1)$$

3) Calcolare l'integrale:

$$I = \int_\Gamma (x+y)\, ds$$



dove $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, è la linea generalmente regolare, chiusa disegnata in figura.

Si osservi che Γ_1 ha equazione:

$$y = x^2 \quad (y = f(x))$$

Dunque il parametro in questione è x. $(0 \le x \le 1)$. Γ_2 ha equazioni:

$$\begin{cases} x = 1 \\ y = y \end{cases} \Rightarrow y \text{ è il parametro } (0 \le y \le 1).$$

Γ_3 ha equazioni: $\begin{cases} y = 0 \\ x = x \end{cases} \quad (0 \le x \le 1)$

$$\Downarrow$$

$$\int_\Gamma (x+y)\,ds = \int_{\Gamma_1} (x+y)\,ds + \int_{\Gamma_2} (x+y)\,ds + \int_{\Gamma_3} (x+y)\,ds$$

Prendiamo:

$$\int_{\Gamma_1} (x+y)\,ds = \int_0^1 (x+y)\sqrt{1+4x^2}\,dx$$

visto che:

$$ds = \sqrt{1+4x^2}\,dx$$

Inoltre:

$$\rightarrow \int_{\Gamma_2} (x+y)\,ds = \int_0^1 (x+y)\sqrt{1}\,dy \qquad \text{con } ds = \sqrt{0+1}\,dy$$

$$\rightarrow \int_{\Gamma_3} (x+y)\,ds = \int_0^1 (x+y)\sqrt{1}\,dx \qquad \text{con } ds = \sqrt{1+0}\,dx$$

visto che su Γ_2 si ha $x=1$, su Γ_3 si ha $y=0$, e su Γ_1 si ha $y=x^2$ si ottiene:

$$\int_\Gamma (x+y)\,ds = \int_0^1 (x+x^2)\sqrt{1+4x^2}\,dx + \int_0^1 (1+y)\,dy + \int_0^1 x\,dx.$$

Siccome:

$$(\Gamma_1): \int_0^1 (x+x^2)\sqrt{1+4x^2}\,dx = \int_0^1 x\sqrt{1+4x^2}\,dx + \int_0^1 x^2\sqrt{1+4x^2}\,dx \Rightarrow \text{Bisogna integrare per sost.}$$

Analizziamo prima il seguente integrale:

$$\int_0^1 x\sqrt{1+4x^2}\,dx$$

Posto:

$$\begin{aligned} y &= 4x^2 \\ y' &= 8x \end{aligned} \Rightarrow \frac{1}{8}\int (1+4x^2)^{\frac{1}{2}}\cdot d(4x^2) = \frac{1}{12}(1+4x^2)^{\frac{3}{2}}$$

Poi:

$$\int_0^1 x^2\sqrt{1+4x^2}\,dx$$

Posto: $2x = \mathrm{Sh}\,t \Rightarrow dx = \frac{1}{2}\mathrm{Ch}\,t\,dt$

$$\Downarrow$$

$$\int x^2\sqrt{1+4x^2}\,dx = \int \frac{1}{4}\mathrm{Sh}^2t\sqrt{1+\mathrm{Sh}^2t}\ \frac{1}{2}\mathrm{Ch}\,t\,dt = \frac{1}{8}\int \mathrm{Sh}^2t\,\mathrm{Ch}^2t\,dt =$$

$$= \frac{1}{32}\int \mathrm{Sh}^2 2t\,dt = \frac{1}{64}\int(\mathrm{Ch}4t-1)\,dt = \frac{1}{64}\left(\frac{1}{4}\mathrm{Sh}4t - t\right) = \frac{1}{64}(\mathrm{Sh}t\,\mathrm{Ch}t($$

$$\mathrm{Ch}^2t+\mathrm{Sh}^2t)-t) = \frac{1}{64}\left[2x(1+4x^2)^{\frac{3}{2}} + 8x^3(1+4x^2)^{\frac{1}{2}} - \log(2x+\sqrt{1+4x^2})\right)$$

Infatti:

- $\sqrt{k+x^2} = \sqrt{k}\,\mathrm{Ch}t \Rightarrow x = \sqrt{k}\,\mathrm{Sh}t \Rightarrow \sqrt{k}=1,\quad x = 2x$

$$\Downarrow$$

$$2x = \sqrt{1}\,\mathrm{Sh}t$$

- $2\mathrm{Sh}x\,\mathrm{Ch}x = \mathrm{Sh}2x \Rightarrow$ se: $4\mathrm{Sh}^2t\,\mathrm{Ch}^2t = \mathrm{Sh}^2 2t \Rightarrow \frac{\mathrm{Sh}^2 2t}{4} = \mathrm{Sh}^2t\,\mathrm{Ch}^2t$

$$\Downarrow$$

$$\frac{1}{8}\cdot\frac{1}{4} = \frac{1}{32}$$

- $\mathrm{Sh}^2t = \frac{\mathrm{Ch}2t-1}{2} \Rightarrow \mathrm{Sh}^2 2t = \mathrm{Ch}4t-1$

Dunque:

$$\int_\Gamma (x+y)\,ds = \left[\frac{1}{12}(1+4x^2)^{\frac{3}{2}} + \frac{1}{32}x(1+4x^2)^{\frac{3}{2}} + \frac{1}{8}x^3(1+4x^2)^{\frac{1}{2}} - \frac{1}{64}\log(2x+\right.$$

$$\left.+\sqrt{1+4x^2})\right]_0^1 + \left[y+\frac{y^2}{2}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{12}5^{\frac{3}{2}} + \frac{1}{32}5^{\frac{3}{2}} + \frac{1}{8}5^{\frac{1}{2}} - \frac{1}{64}\log(2+\sqrt{5}) +$$

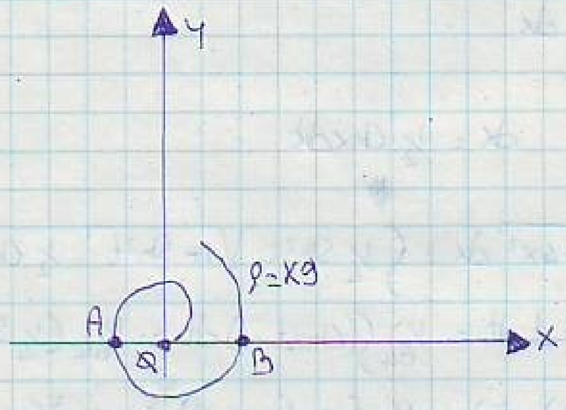
$$+(-)\frac{1}{12} + \frac{3}{2} + \frac{1}{2} = \frac{67}{96}\sqrt{5} - \frac{1}{64}\log(2+\sqrt{5}) + \frac{23}{12}.$$

4) Calcolare l'integrale:

$$I = \int_{p} \sqrt{x^2+y^2}\, ds$$

dove è l'arco $\widehat{OAB}$ disegnato in figura, appartenente alla spirale di archimede di equazione:

$$\underline{\rho = k\vartheta} \qquad (k = \text{cost.} > 0)$$



Utilizzando le coordinate polari si ottiene: $(0 \le \vartheta \le 2\pi)$

$$\begin{cases} x = \rho\cos\vartheta = k\vartheta\cos\vartheta \\ y = \rho\sin\vartheta = k\vartheta\sin\vartheta \end{cases}$$

$$\Downarrow$$

$$\boxed{ds = \sqrt{\rho^2(\vartheta) + \rho'^2(\vartheta)}}$$ → nei casi in cui si utilizzano le variabili polari.

$$ds = \sqrt{k^2\vartheta^2 + k^2} \Rightarrow ds = \sqrt{k^2(\vartheta^2+1)} = k\sqrt{\vartheta^2+1}\, d\vartheta$$

$$\Downarrow$$

$$I = \int_0^{2\pi} k\vartheta \cdot k\sqrt{\vartheta^2+1}\, d\vartheta = k^2 \int_0^{2\pi} \vartheta\sqrt{1+\vartheta^2}\, d\vartheta$$

Sostituzione:

$$t = 1+\vartheta^2 \Rightarrow t-1 = \vartheta^2 \Rightarrow \vartheta = \sqrt{t-1} \Rightarrow d\vartheta = \frac{1}{2\sqrt{t-1}}\, dt$$

Dunque:

$$k^2 \int_0^{2\pi} \sqrt{t}\sqrt{t-1}\, \frac{1}{2\sqrt{t-1}}\, dt = \frac{k^2}{2}\int_0^{2\pi} \sqrt{t}\, dt = \frac{k^2}{3} t^{3/2} + K_c$$ → costante arbitraria

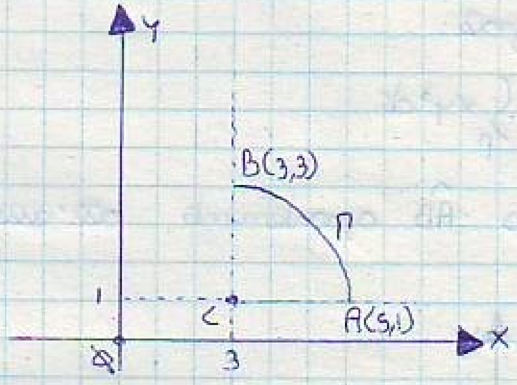
Dunque:

$$\frac{K^2}{3}\left[(1+\vartheta^2)^{3/2}\right]_0^{2\pi} = \frac{K^2}{3}\left((1+4\pi^2)^{3/2}-1\right)$$

5) Calcolare l'integrale:

$$I=\int_\Gamma xy\,ds$$

dove Γ è l'arco della circonferenza di centro $C(3,1)$ e raggio $R=2$, disegnato in figura.

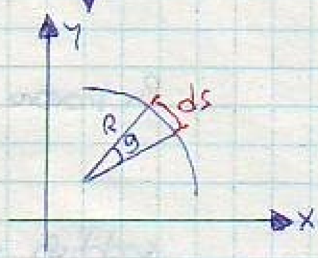


Equazioni parametriche di Γ:

$$\begin{cases} x=3+2\cos\vartheta \\ y=1+2\sin\vartheta \end{cases} \qquad \left(0\le\vartheta\le\frac{\pi}{2}\right)$$

$$\Downarrow$$

$$ds=\sqrt{x'^2(\vartheta)+y'^2(\vartheta)}\,d\vartheta = 2\,d\vartheta \rightarrow \boxed{ds=R\,d\vartheta}$$



$$\Downarrow$$

$$I=\int_\Gamma xy\,ds=\int_0^{\pi/2}(3+2\cos\vartheta)(1+2\sin\vartheta)\,2\,d\vartheta=$$

$$=2\int_0^{\pi/2}(3+6\sin\vartheta+2\cos\vartheta+4\sin\vartheta\cos\vartheta)\,d\vartheta=$$

$$=2\int_0^{\pi/2}3\,d\vartheta+12\int_0^{\pi/2}\sin\vartheta\,d\vartheta+2\int 2\cos\vartheta+$$

$$+8\int_0^{\pi/2}\sin\vartheta\cos\vartheta\,d\vartheta=\left[6\vartheta-12\cos\vartheta+4\sin\vartheta\right]_0^{\pi/2}+8\int_0^{\pi/2}\sin\vartheta\cos\vartheta\,d\vartheta$$

Quest'ultimo integrale si calcola così:

$$\int\sin\vartheta\cos\vartheta\,d\vartheta=\ldots\quad \sin\vartheta=t \Rightarrow dt=\cos\vartheta\,d\vartheta$$

$$\Downarrow$$

$$\int t\,dt=\frac{t^2}{2}+c$$

Quindi:

$$\int \operatorname{sen}\vartheta \cos\vartheta \, d\vartheta = \frac{\operatorname{sen}^2\vartheta}{2} + c$$

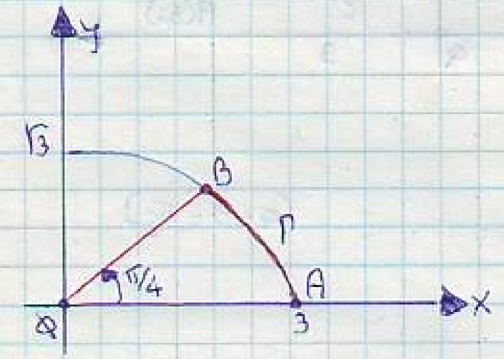
$$\Downarrow$$

$$4\left[\operatorname{sen}\vartheta\right]_0^{2\pi} = 0 \Rightarrow I = 3\pi - 8$$

6) Calcolare l'integrale:

$$I = \int_\Gamma xy^3 \, ds$$

dove Γ è l'arco $\widehat{AB}$ appartenente all'ellisse di semiassi 3 e $\sqrt{3}$ disegnato in figura.



Equazioni parametriche:

$$\begin{cases} x = 3\cos\vartheta \\ y = \sqrt{3}\operatorname{sen}\vartheta \end{cases} \qquad 0 \le \vartheta \le \vartheta_1$$

Per trovare ϑ_1 si osservi che:

$$B(\bar{x}, \bar{y}) = B(3\cos\vartheta_1, \sqrt{3}\operatorname{sen}\vartheta_1)$$

soddisfa la condizione:

$$\operatorname{tg}\frac{\pi}{4} = 1 = \frac{\bar{y}}{\bar{x}} \Rightarrow 1 = \frac{\sqrt{3}\operatorname{sen}\vartheta_1}{3\cos\vartheta_1} \Rightarrow \operatorname{Tg}\vartheta_1 = \sqrt{3}$$

$$\Downarrow$$

$$\vartheta_1 = \pi/3$$

$$ds = \sqrt{9\operatorname{sen}^2\vartheta + 3\cos^2\vartheta} \, d\vartheta = \sqrt{3 + 6\operatorname{sen}^2\vartheta} \, d\vartheta$$

$$\Downarrow$$

$$I = \int_\Gamma xy^3 \, ds = \int_0^{\pi/3} 3\cos\vartheta \; 3\sqrt{3}\operatorname{sen}^3\vartheta \sqrt{3 + 6\operatorname{sen}^2\vartheta} \, d\vartheta = 9\sqrt{3}\int_0^{\pi/3} \operatorname{sen}^2\vartheta \sqrt{3 + 6\operatorname{sen}^2\vartheta} \operatorname{sen}\vartheta \cos\vartheta \, d\vartheta = \dots$$

Posto: $\sqrt{3+6\,\text{sen}^2\vartheta} = t \Rightarrow t^2 = 3+6\,\text{sen}^2\vartheta \Rightarrow t^2-3 = 6\,\text{sen}^2\vartheta \Rightarrow \frac{t^2-3}{6} = \text{sen}\,\vartheta$

Quindi:

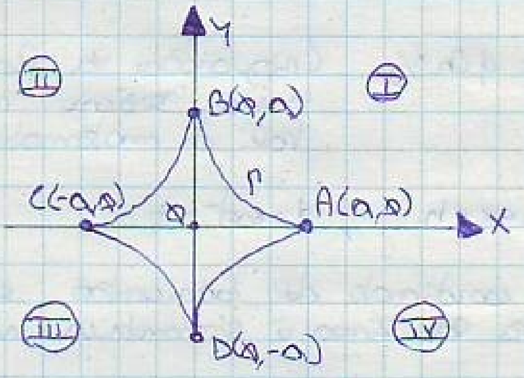
$$\text{sen}^2\vartheta = \text{sen}\,\vartheta\,\text{sen}\,\vartheta \Rightarrow 2\,\text{sen}\,\vartheta\cos\vartheta\,d\vartheta = \tfrac{1}{3}t\,dt \Rightarrow \text{sen}\,\vartheta\cos\vartheta\,d\vartheta = \tfrac{1}{6}t\,dt$$

$$I = \int_\gamma xy^3\,ds = 9\sqrt{3}\int_{\sqrt{3}}^{\sqrt{15/2}} \frac{t^2-3}{6}\,t\,\tfrac{1}{6}t\,dt = \tfrac{\sqrt{3}}{4}\int_{\sqrt{3}}^{\sqrt{15/2}}(t^4-3t^2)\,dt = \tfrac{\sqrt{3}}{4}\left[\frac{t^5}{5}-t^3\right]_{\sqrt{3}}^{\sqrt{15/2}}$$

$$= \tfrac{\sqrt{3}}{20}\left[t^3(t^2-5)\right]_{\sqrt{3}}^{\sqrt{15/2}} = \tfrac{\sqrt{3}}{20}\left(\tfrac{15}{2}\sqrt{\tfrac{15}{2}}\left(\tfrac{15}{2}-5\right)+6\sqrt{3}\right) = \tfrac{45}{16}\sqrt{\tfrac{5}{2}}+\tfrac{9}{10}$$

7) Calcolare la lunghezza ℓ dell'asteroide disegnato in figura.
Le equazioni parametriche dell'asteroide sono:

$$\begin{cases} x = a\cos^3\vartheta \\ y = a\,\text{sen}^3\vartheta \end{cases} \qquad (0 \le \vartheta \le 2\pi)$$



Siccome la figura è simmetrica rispetto ai assi x,y basta calcolare la lunghezza di γ nel I quadrante.

$$ds = \sqrt{9a^2(\cos^4\vartheta\,\text{sen}^2\vartheta + \text{sen}^4\vartheta\cos^2\vartheta)}\,d\vartheta = 3a\,|\text{sen}\,\vartheta\cos\vartheta|\,d\vartheta.$$

Per $0 \le \vartheta \le \pi/2 \Rightarrow |\text{sen}\,\vartheta\cos\vartheta| = \text{sen}\,\vartheta\cos\vartheta$

$$\ell = 12a\int_0^{\pi/2}\text{sen}\,\vartheta\cos\vartheta = 6a\left[\text{sen}^2\vartheta\right]_0^{\pi/2} = 6a$$

$3a \cdot 4$
↳ numero "lati".